# Latest on Linear Sketches for Large Graphs: Lots of Problems, Little Space, and Loads of Handwaving

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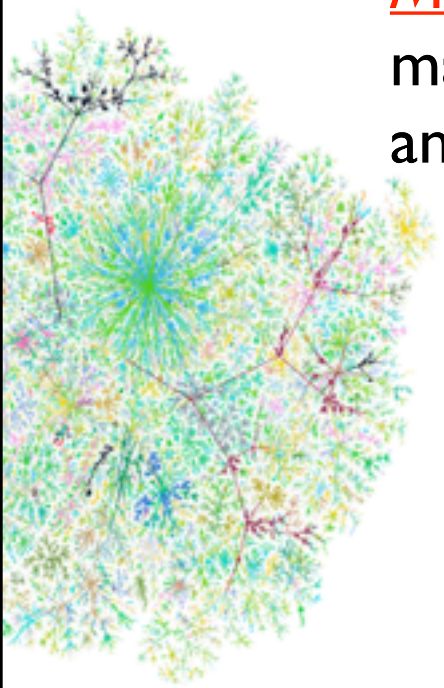
**Vertex Connectivity and Sparsification** — Guha, McGregor, Tench [PODS 2015]

**Densest Subgraphs** — McGregor, Tench, Vorotnikova, Vu [MFCS 2015]

**Matching, Vertex Cover, Hitting Set**
Chitnis, Cormode, Esfandiari, Hajiaghayi, McGregor, Monemizadeh, Vorotnikova [TBA 2016]



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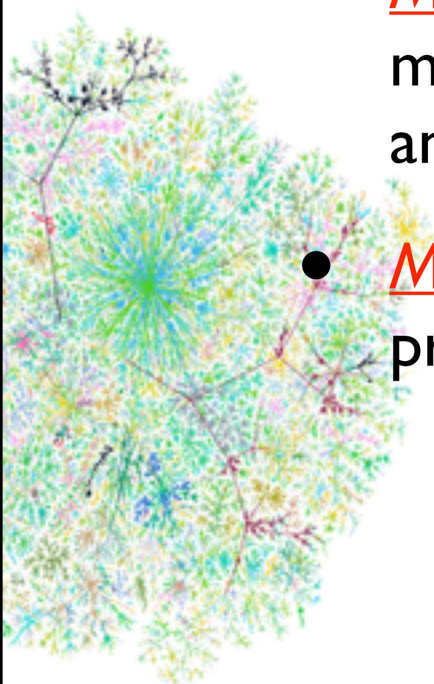

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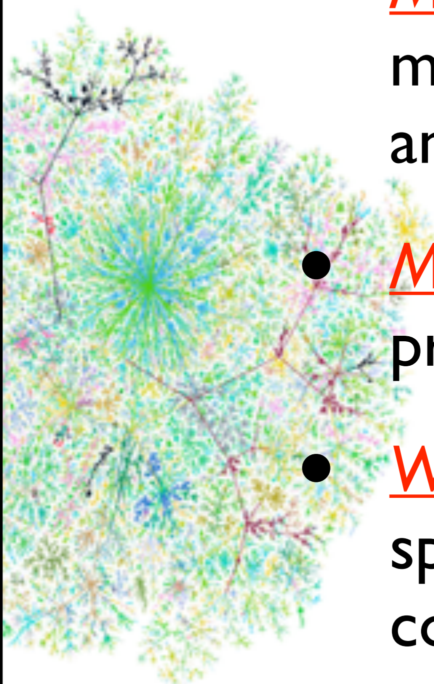

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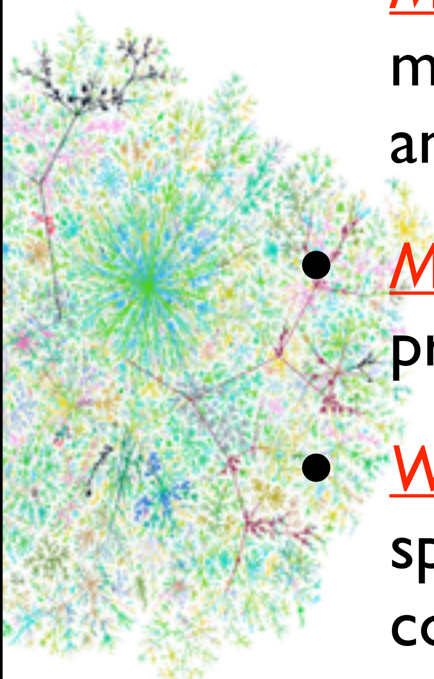

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Graph Stream Algorithms: A Survey

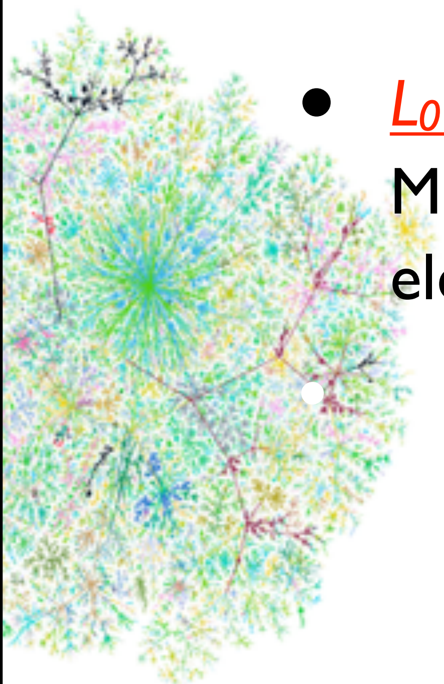
Andrew McGregor

ABSTRACT

1. INTRODUCTION

**Graph Streaming Survey**

**McGregor [SIGMOD Record 2014]**



- *$L_0$ Sampling Primitive* There's a distribution over matrices $M \in \Re^{polylog(N) \times N}$ such that for any $x \in \Re^N$, a random non-zero element of x can be reconstructed from Mx whp.

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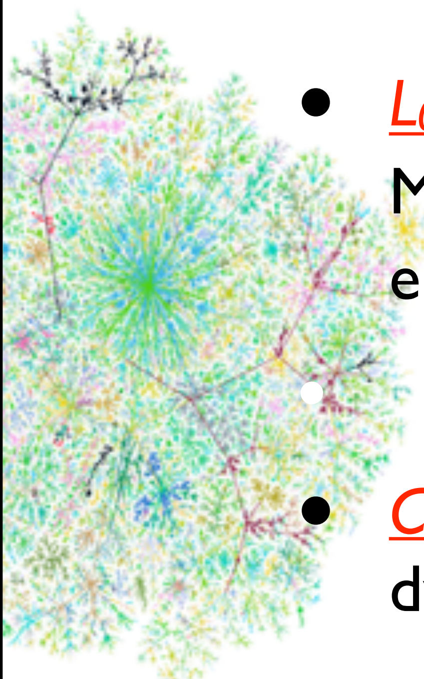

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- *Corollary* Can sample a uniform edge from a graph in the dynamic graph stream model using O(polylog n) bits of space.

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see also *Mitzenmacher et al.* [KDD 2015], *Esfandiari et al.* [ArXiv 2015]

*What other types of sampling are there that a) are useful for solving graph problems and b) can be supported on dynamic graph streams?*

*I*. Graph Matching
via **SNAPE** Sampling


*II*. Graph Connectivity
via **DEALS** Sampling

# Graph Matchings

- *1st Result* If max matching has size $\leq k$, can find optimal matching in dynamic stream model using $\tilde{O}(k^2)$ space.
  - *Optimal & Simple.* Extends to hypergraph matching, vertex cover, hitting set… but gets a lot more complicated.
  - *Basic Idea:* "SNAPE" sampling primitive.

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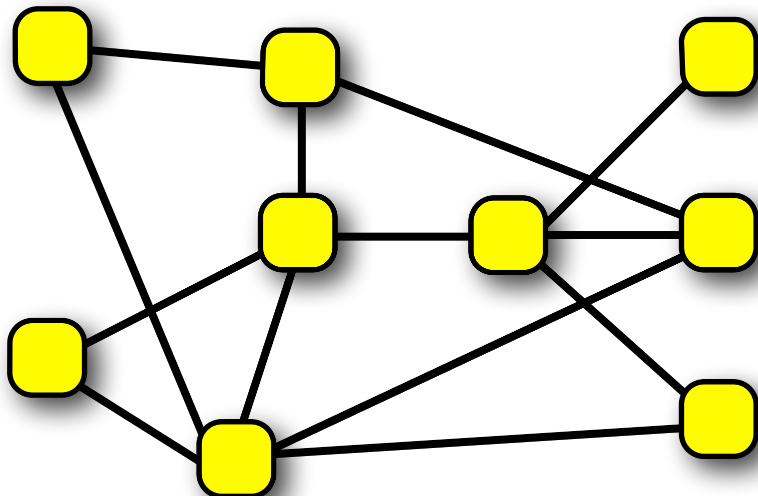
  - *Optimal & Simple.* Extends to hypergraph matching, vertex cover, hitting set… but gets a lot more complicated.
  - *Basic Idea:* "SNAPE" sampling primitive.
- *2nd Result* If max matching has size $\geq k$, can find matching of size $\Omega(k/t)$ in the dynamic stream model using $\tilde{O}(k^2/t^3)$ space.
  - *Application:* Guessing k gives O(t)-approx for max matching using $\tilde{O}(n^2/t^3)$ space. This is also optimal; see Sanjeev's talk.

# SNAPE Sampling

## Sample-Nodes-And-Pick-Edge



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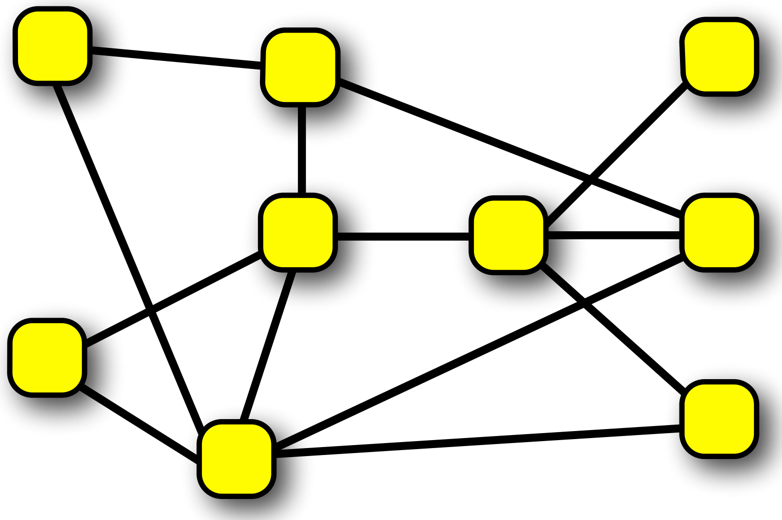

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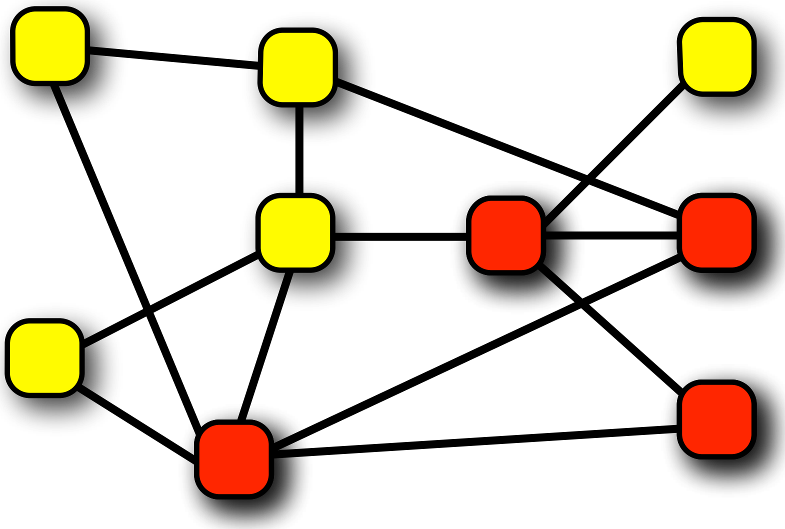

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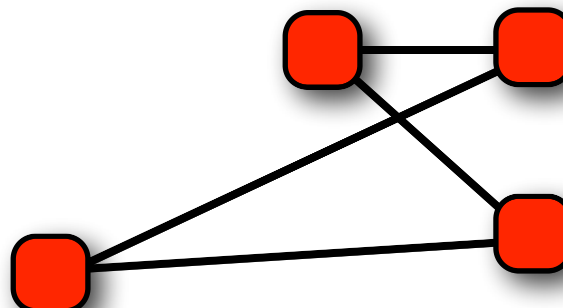

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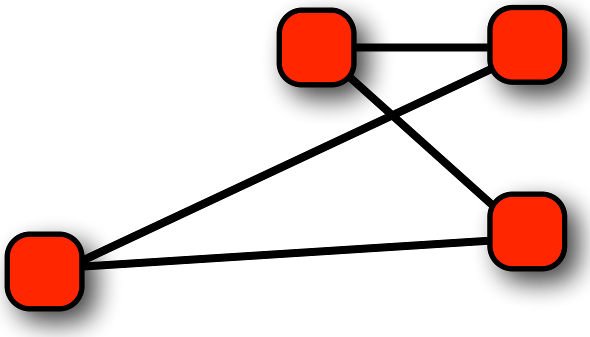

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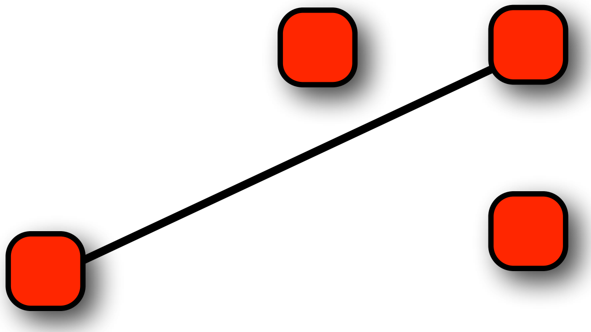

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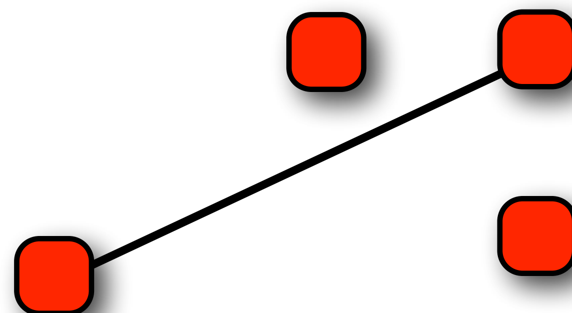
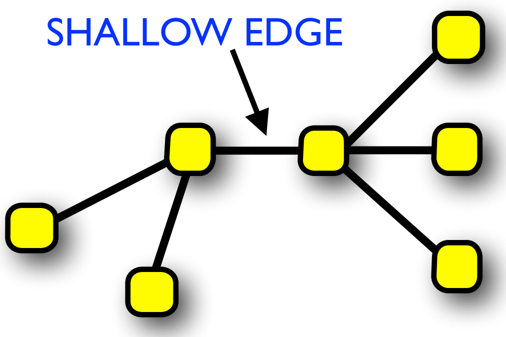
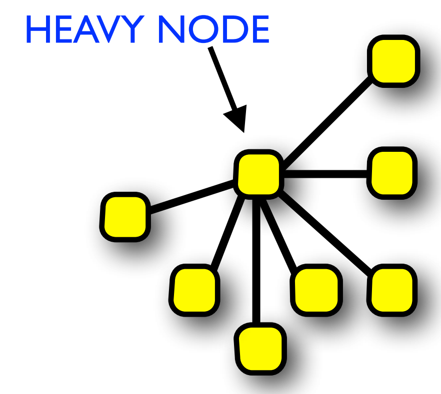


- SAMPLE each node with prob. $\Theta(1/k)$ and DELETE the rest
- RETURN a random edge amongst those that remain. If no edges remain, return NULL.
- *Theorem* If G has max matching size $\leq k$ then $O(k^2 \log k)$ SNAPE samples will include a max matching from G.

# *Small Matching Analysis: Basic Idea*

- Let G have max matching of size $\leq$k. Say node is heavy if degree is $\geq$10k and edge is shallow if both endpoints aren't heavy.



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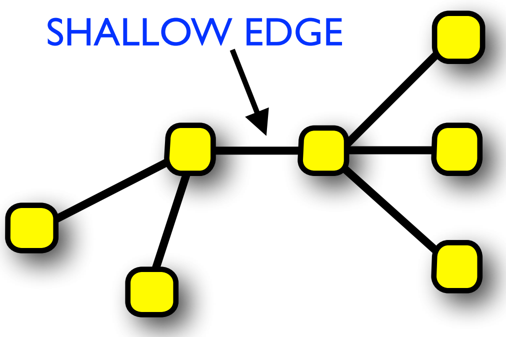


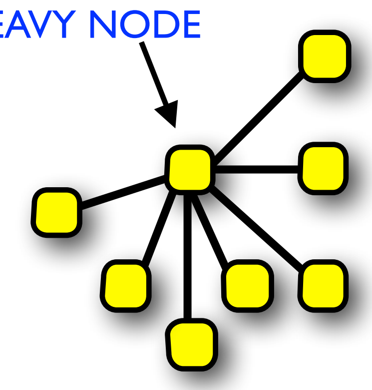

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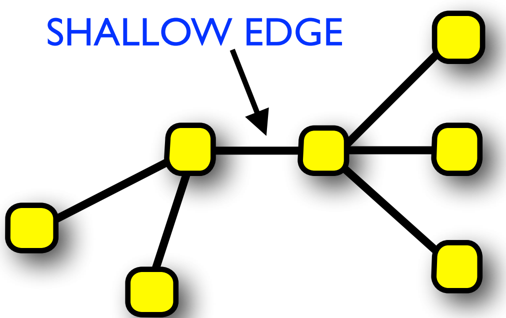


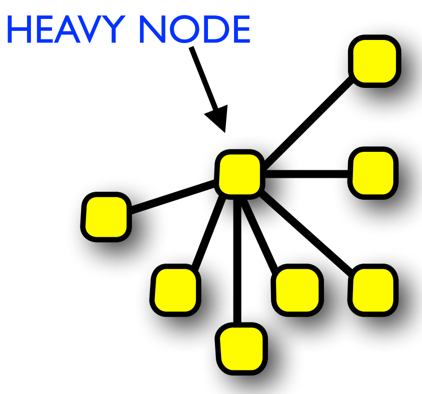


- *Lemma* Let G' contains a max matching of G if:
  - G' includes all shallow edges in G.
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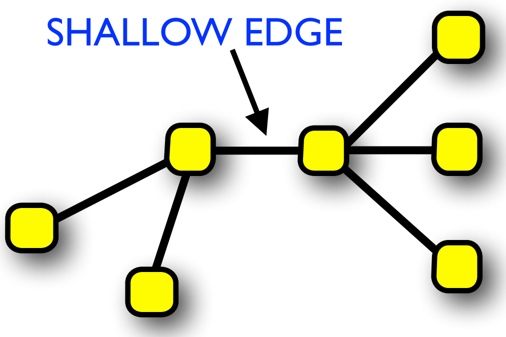


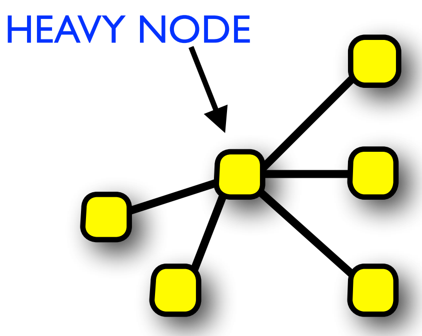


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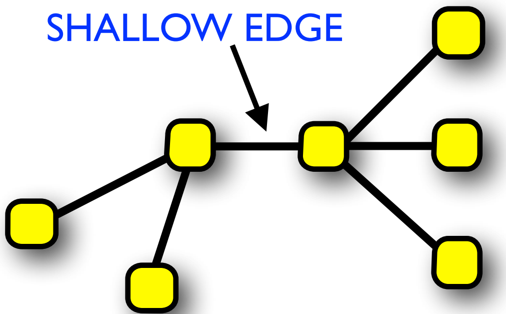


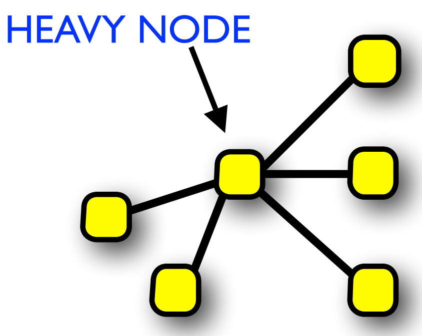


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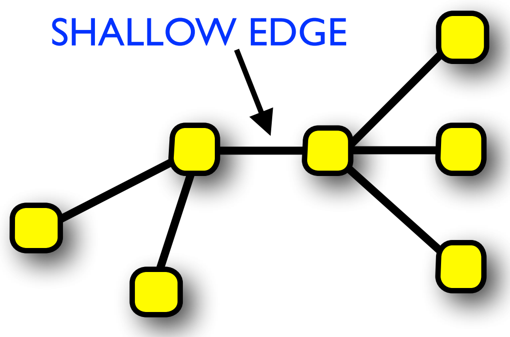


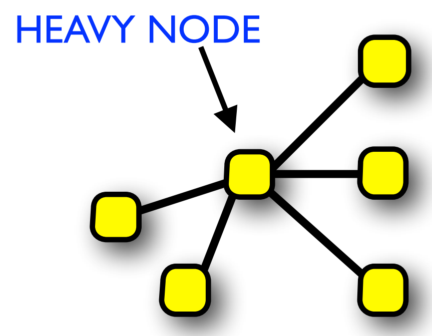


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- *Useful Fact* G has a vertex cover W of size at most 2k.

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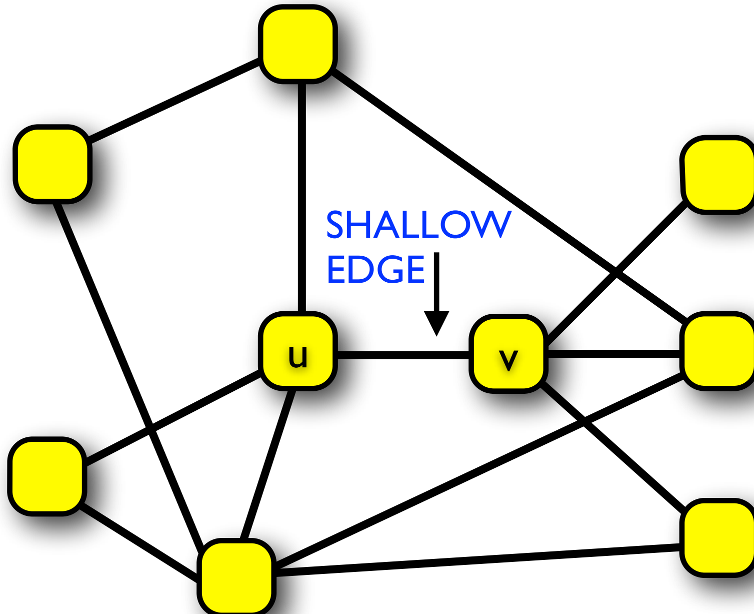

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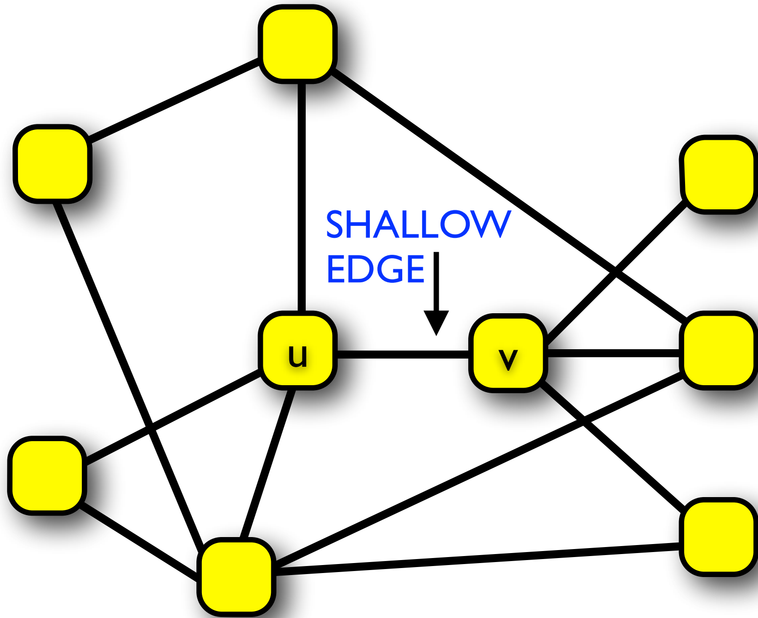


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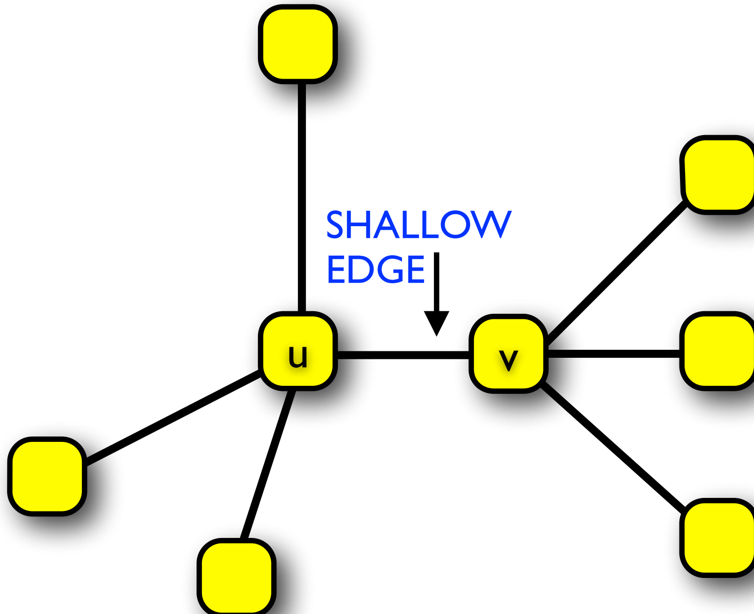


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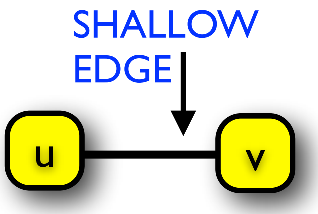


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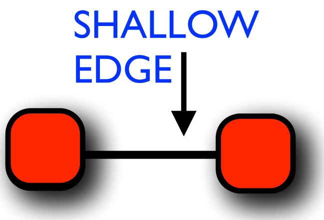


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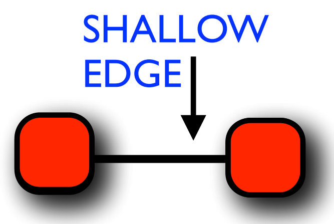


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- Hence, if uv is shallow:

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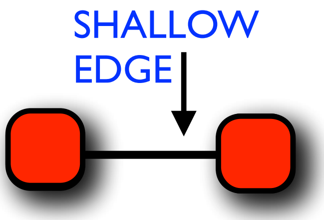


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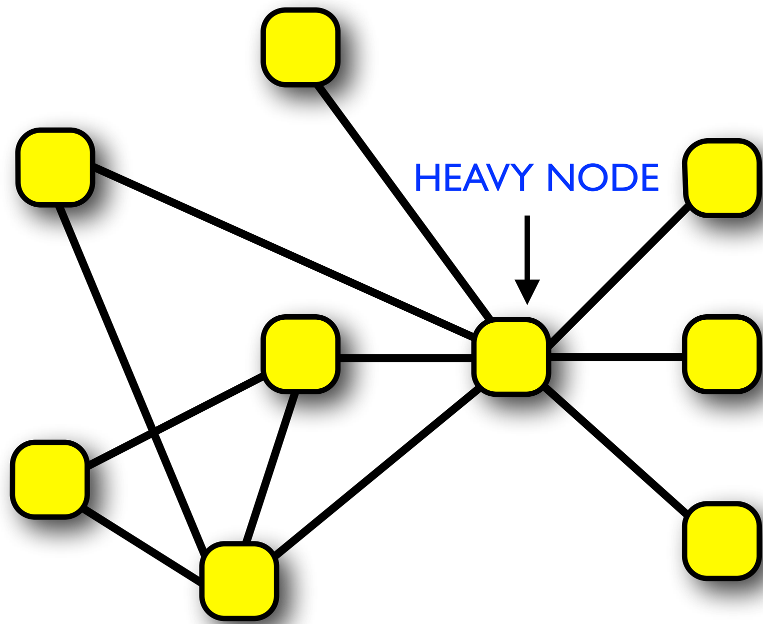
- After $O(k^2 \log k)$ repetitions, have sampled edge uv whp.

# *Small Matching Analysis: Edges on Heavy Nodes*



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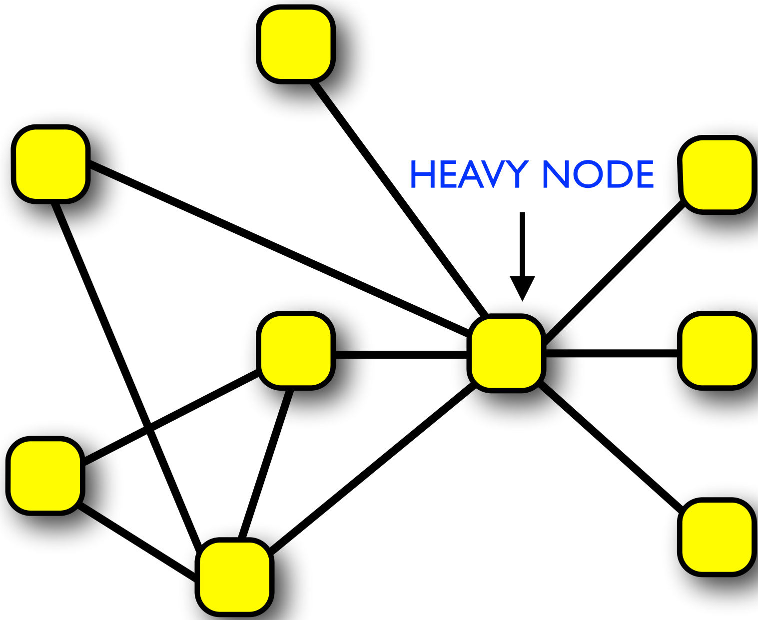


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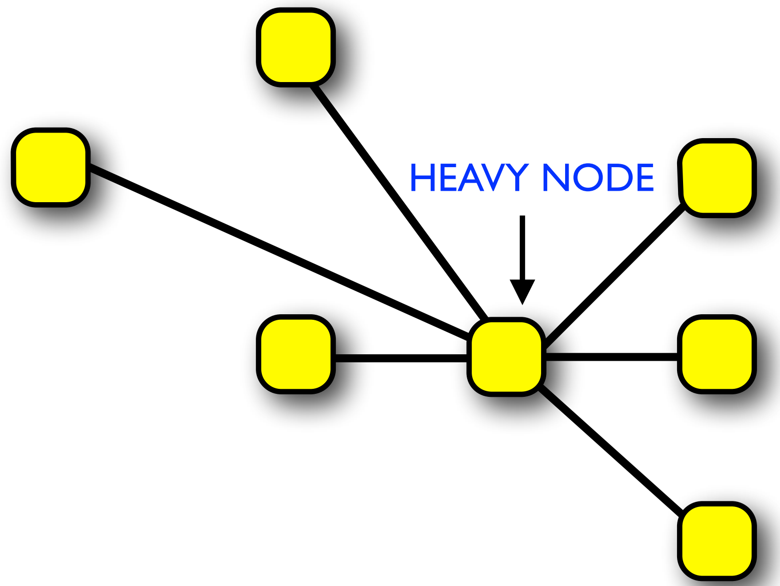


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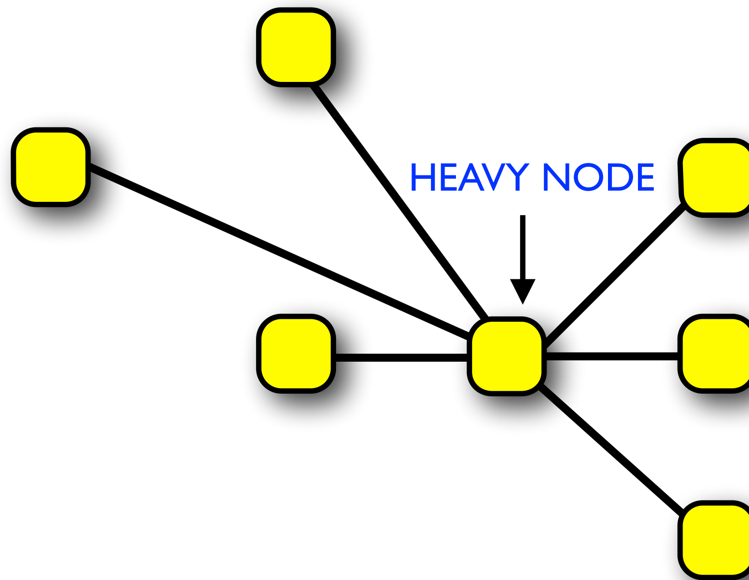


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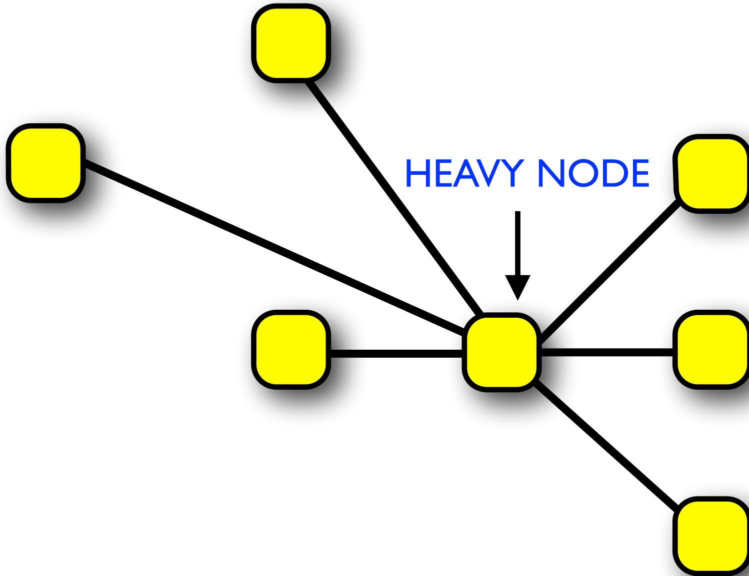


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$$\Pr[\text{edge incident to u is sampled}] \geq p(1-p)^{|W|} = \Omega(k^{-1})$$

- After $O(k^2 \log k)$ repetitions, have sampled 5k edges on u.

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*I*. Graph Matching
via **SNAPE** Sampling


*II*. Graph Connectivity
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- *3rd Result* $(1+\varepsilon)$-approx every cut using $\tilde{O}(\varepsilon^{-2}n)$ space.
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  - Hypergraph Sparsifiers: Extends *Kogan, Krauthgamer* [ITCS 2015]


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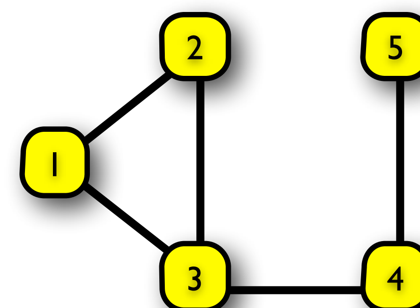


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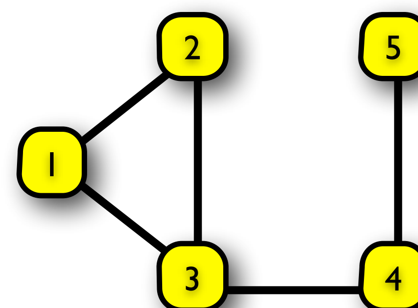


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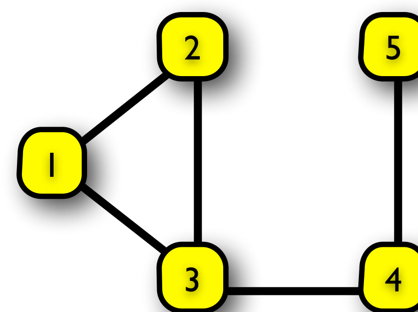


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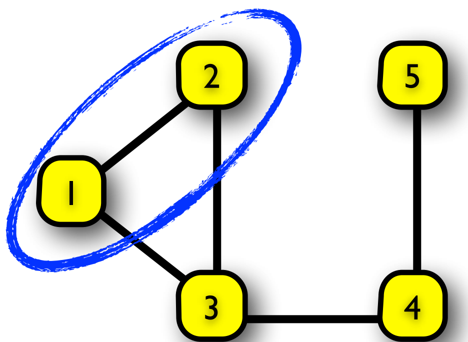


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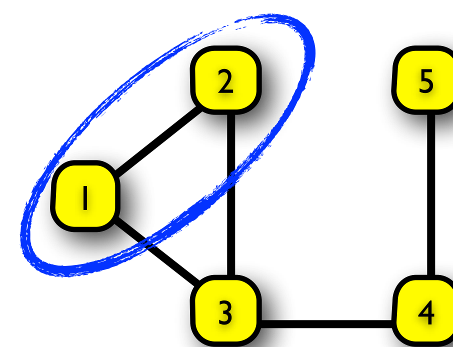


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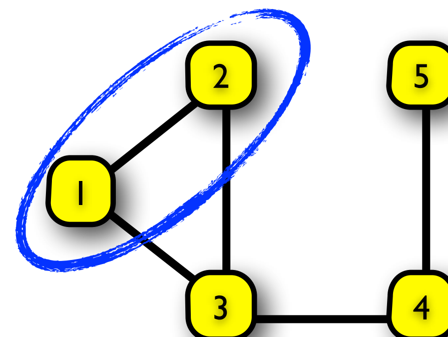


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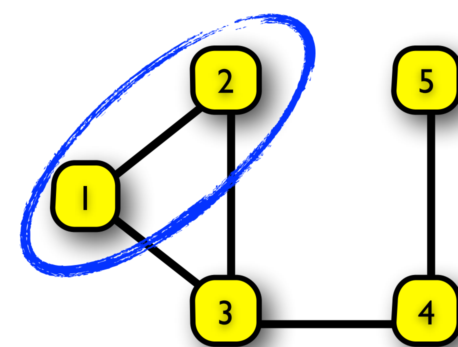


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- *Application* Find spanning trees and edges in light cuts.

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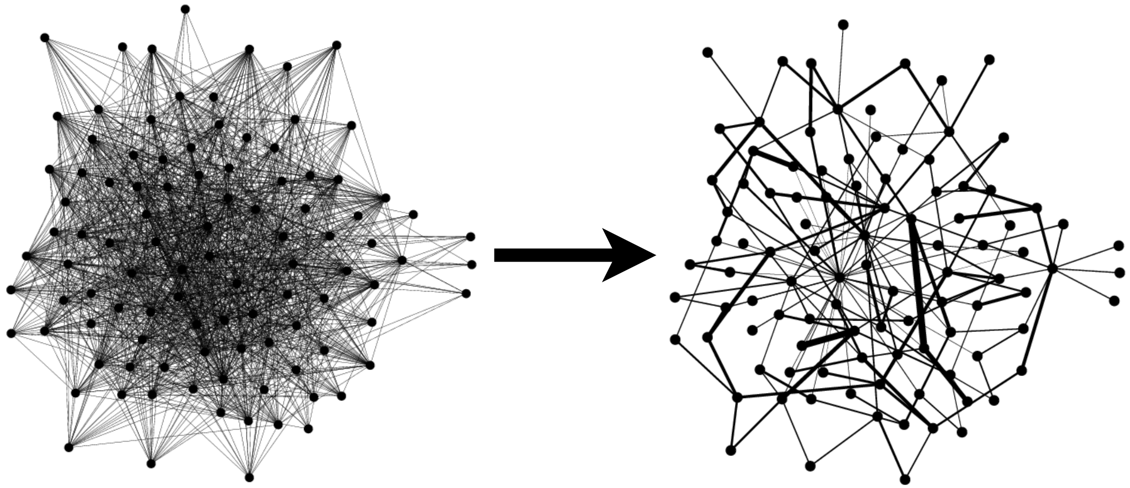

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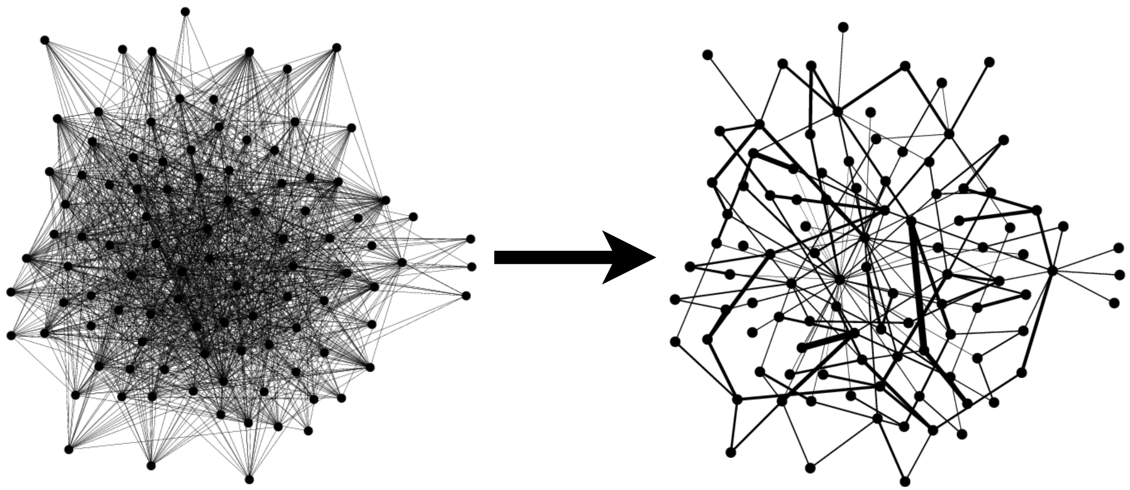

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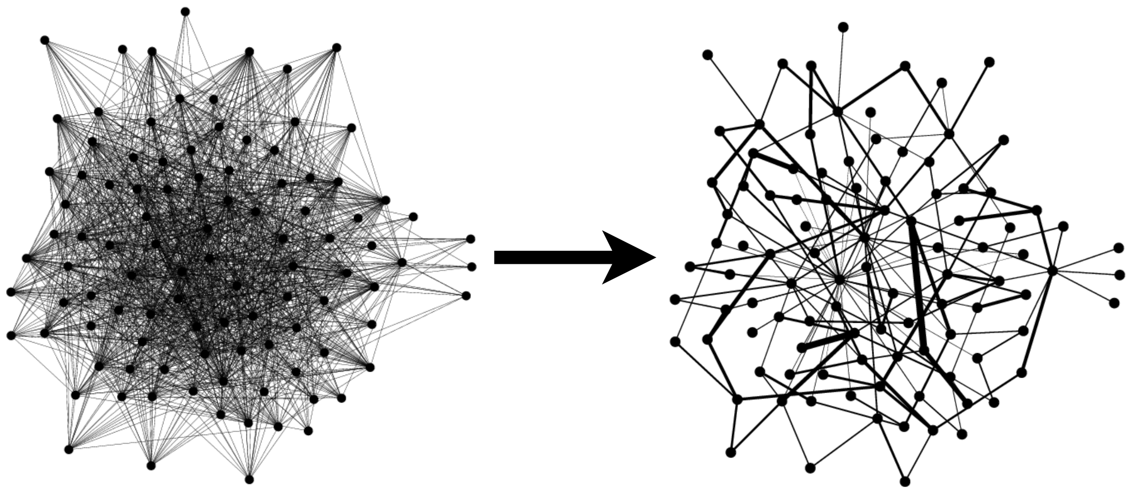

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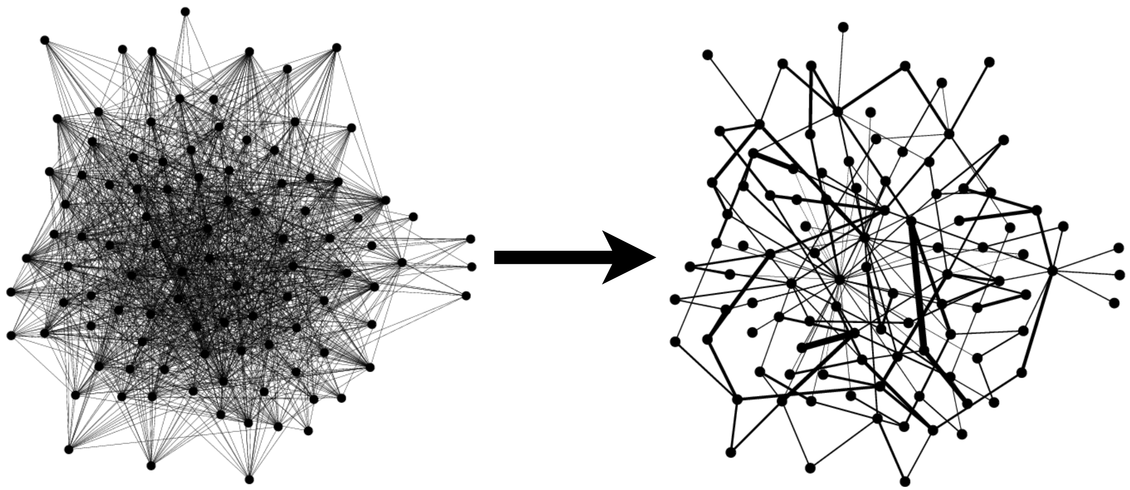

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  - Recurse $O(\log n)$ times in parallel until we have sparse graph.

*Thanks!*

Graph Streaming Survey McGregor [SIGMOD Record 2014]

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